

Higher Derivative Gravities and Negative Entropy

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We investigate the black hole solutions in the R^2 -gravity, where the action contains the square of the curvature. In case that the action does not contain the square of the Riemann tensor and in case that the R^2 -terms are the Gauss-Bonnet (GB) combination, we find exact solutions. We investigate the thermodynamics of these theories and find the Hawking-Page like phase transition, which is the phase transition between the black hole (BH) spacetime and the pure anti-deSitter (AdS) spacetime. From the viewpoint of the AdS/CFT correspondence, such a phase transition may correspond to thermal transition of dual CFT.

An interesting feature of R^2 -gravity is the possibility of the negative (or zero) dS (or AdS) BH entropy, which depends on the parameters of the R^2 -terms. We speculate that the appearance of negative entropy may indicate a new type instability where a transition between dS (AdS) BH with negative entropy and AdS (dS) BH with positive entropy occurs.

We also apply the GB gravity to the brane cosmology, where the brane moves in the bulk AdS BH spacetime. By investigating the FRW-like equation, which describes the motion of the brane, we find the behavior of the matter on the brane. When the radius of the brane is large, the matter fields behave as CFT but when the radius is small, the brane universe behaves as the universe with dust or curvature dominant universe, depending on the parameters.

1. Introduction

The Einstein-Hilbert action includes only scalar curvature

$$S_{\text{Einstein}} = \int d^{d+1}x \sqrt{-g} \left\{ \frac{1}{\kappa^2} R - \Lambda \right\} + S_{\text{matter}} , \quad (1)$$

Here S_{matter} is the action of matter fields. On the other hand the action of R^2 -gravity includes the square of the curvatures

$$S = \int d^{d+1}x \sqrt{-g} \{ a R^2 + b R_{\mu\nu} R^{\mu\nu} + c R_{\mu\nu\xi\sigma} R^{\mu\nu\xi\sigma} \} + S_{\text{Einstein}} . \quad (2)$$

One of the reasons why we consider the R^2 -gravity is because the limit of $\alpha' \rightarrow 0$ limit in string theory corresponds the Einstein gravity but if we include the first order correction with respect to α' , we obtain R^2 -gravity. From the viewpoint of the AdS/CFT correspondence [1] (or recent dS/CFT correspondence[2, 3, 4]), α' correction corresponds to $1/N$ corrections in the field theory side. The equations of motion of the R^2 -gravity, which correspond to the Einstein equation has the following form:

$$0 = \frac{1}{2} g^{\mu\nu} (a R^2 + b R_{\mu\nu} R^{\mu\nu} + c R_{\mu\nu\xi\sigma} R^{\mu\nu\xi\sigma} + \frac{1}{\kappa^2} R - \Lambda) + a (-2 R R^{\mu\nu} + \nabla^\mu \nabla^\nu R + \nabla^\nu \nabla^\mu R - 2 g^{\mu\nu} \nabla_\rho \nabla^\rho R)$$

$$+ b (-2 R^\mu_\rho R^{\nu\rho} + \nabla_\rho \nabla^\mu R^{\rho\nu} + \nabla_\rho \nabla^\nu R^{\rho\mu} - \square R^{\mu\nu} - g^{\mu\nu} \nabla^\rho \nabla^\sigma R_{\rho\sigma}) + c (-2 R^{\mu\rho\sigma\tau} R^\nu_{\rho\sigma\tau} - 2 \nabla_\rho \nabla_\sigma R^{\mu\rho\nu\sigma} - 2 \nabla_\rho \nabla_\sigma R^{\nu\rho\mu\sigma}) - \frac{1}{\kappa^2} R^{\mu\nu} + T^{\mu\nu} . \quad (3)$$

Here $T^{\mu\nu}$ is energy-momentum tensor of matter: $T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g_{\mu\nu}}$. The above equation (3) is very complicated but there are several solvable cases. One is $c = 0$ case, where the Riemann tensor $R_{\nu\rho\mu\sigma}$ does not appear in the equation of motion, then the solutions can be constructed from the solutions of the Einstein equation. The other is the Gauss-Bonnet (GB) invariant combination ($a = c$, $b = -4c$) case. Of course, the GB term is trivial (topological) in 4 dimensions but non-trivial when the dimensions of the spacetime are more than 4.

We find the black hole (BH) solutions for the $c = 0$ case in Section 2. and for the Gauss-Bonnet combination case in Section 3.. By using the obtained solution, we investigate the thermodynamics of the R^2 -gravity in Section 4. and we find the Hawking-Page like phase transition, which is the phase transition between the black hole spacetime and the pure AdS spacetime. From the viewpoint of the AdS/CFT correspondence, such a phase transition may be that of the field theory when we include $1/N$ corrections. An interesting feature of higher derivative gravity is the possibility of the negative (or zero) dS (or AdS) BH entropy which depends on the parameters of higher derivative terms. In Section 5., we investigate the problem of the negative entropy and we speculate that the appearance of negative entropy may indicate a new type instability where a transition

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between dS (AdS) BH with negative entropy and AdS (dS) BH with positive entropy occurs. In Section 5., we apply the GB gravity to the brane cosmology, where the brane moves in the bulk AdS black hole spacetime. By investigating the FRW-like equation, which describes the motion of the brane, we find the behavior of the matter fields on the brane. When the radius of the brane is large, the matter fields behave as CFT but when the radius is small, the brane universe behaves as the universe with dust or curvature dominant universe, depending on the parameters. The last section is devoted to the summary.

This report is mainly based on [5, 6, 7, 8, 9, 13, 14].

2. Solutions of $c = 0$ case

We first consider the solutions of $c = 0$ case. The $D = d + 1$ dimensional Einstein equation without matter is, of course, given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{\kappa^2}{2}\Lambda g_{\mu\nu}. \quad (4)$$

Then by multiplying $g^{\mu\nu}$, we find the scalar curvature is constant $\frac{d-1}{2}R = \frac{d+1}{2}\kappa^2\Lambda$ and the Ricci tensor is proportional to the metric $R_{\mu\nu} = \frac{d}{2(d-1)}\kappa^2\Lambda g_{\mu\nu}$ and therefore covariantly constant $\nabla_\rho R_{\mu\nu} = 0$ due to the condition of the metricity $\nabla_\rho g_{\mu\nu} = 0$. We now apply this results to the R^2 -gravity with $c = 0$.

By assuming

$$R = -\frac{d(d+1)}{l^2}, \quad R_{\mu\nu} = -\frac{d}{l^2}g_{\mu\nu}, \quad (5)$$

and substituting these expressions of the curvatures to Eq.(3) with $c = 0$, we obtain

$$0 = \frac{d^2(d+1)(d-3)a}{l^4} + \frac{d^2(d-3)b}{l^4} - \frac{d(d-1)}{\kappa^2 l^2} - \Lambda, \quad (6)$$

from which l^2 can be determined. Then if we choose $\frac{1}{l^2} = -\frac{d}{2(d-1)}\kappa^2\Lambda$ in the Einstein equation (4), we can construct a solution of R^2 -gravity (with $c = 0$) from the solution of the vacuum Einstein equation.

3. Solutions in the Gauss-Bonnet combination case

In this section, we consider the solutions in the Gauss-Bonnet combination.

By using Bianchi identity

$$\nabla_\mu R^\lambda_{\sigma\nu\rho} + \nabla_\rho R^\lambda_{\sigma\mu\nu} + \nabla_\nu R^\lambda_{\sigma\rho\mu} = 0, \quad (7)$$

we obtain

$$\begin{aligned} & \nabla_\rho \nabla_\sigma R^{\mu\rho\nu\sigma} \\ &= \square R^{\mu\nu} - \frac{1}{2}\nabla^\mu \nabla^\nu R + R^{\mu\rho\nu\sigma} R_{\rho\sigma} - R^\mu_\rho R^{\nu\rho} \\ & \nabla_\rho \nabla^\mu R^{\rho\nu} + \nabla_\rho \nabla^\nu R^{\rho\mu} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}(\nabla^\mu \nabla^\nu R + \nabla^\nu \nabla^\mu R) \\ & \quad - 2R^{\mu\rho\nu\sigma} R_{\rho\sigma} + 2R^\mu_\rho R^{\nu\rho}, \\ & \nabla_\rho \nabla_\sigma R^{\rho\sigma} = \frac{1}{2}\square R. \end{aligned} \quad (8)$$

By using the above equations (8), we can rewrite the equations of motion (3) as follows:

$$\begin{aligned} 0 &= \frac{1}{2}g^{\mu\nu} \left(aR^2 + bR_{\mu\nu}R^{\mu\nu} + cR_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma} \right. \\ & \quad \left. + \frac{1}{\kappa^2}R - \Lambda \right) \\ & \quad + a(-2RR^{\mu\nu} + \nabla^\mu \nabla^\nu R + \nabla^\nu \nabla^\mu R - 2g^{\mu\nu}\square R) \\ & \quad + b \left\{ \frac{1}{2}(\nabla^\mu \nabla^\nu R + \nabla^\nu \nabla^\mu R) - 2R^{\mu\rho\nu\sigma}R_{\rho\sigma} \right. \\ & \quad \left. - \square R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\square R \right\} \\ & \quad + c(-2R^{\mu\rho\sigma\tau}R^\nu_{\rho\sigma\tau} - 4\square R^{\mu\nu} + \nabla^\mu \nabla^\nu R \\ & \quad + \nabla^\nu \nabla^\mu R - 4R^{\mu\rho\nu\sigma}R_{\rho\sigma} + 4R^\mu_\rho R^{\nu\rho}) \\ & \quad - \frac{1}{\kappa^2}R^{\mu\nu} - T^{\mu\nu}_{\text{matter}}. \end{aligned} \quad (9)$$

When the coefficients a , b , c are Gauss-Bonnet combination: $a = c$ and $b = -4c$, one finds

$$\begin{aligned} 0 &= \frac{1}{2}g^{\mu\nu} \left\{ c(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}) \right. \\ & \quad \left. + \frac{1}{\kappa^2}R - \Lambda \right\} + c(-2RR^{\mu\nu} + 4R^\mu_\rho R^{\nu\rho} \\ & \quad + 4R^{\mu\rho\nu\sigma}R_{\rho\sigma} - 2R^{\mu\rho\sigma\tau}R^\nu_{\rho\sigma\tau}) \\ & \quad - \frac{1}{\kappa^2}R^{\mu\nu} + T^{\mu\nu}. \end{aligned} \quad (10)$$

Since the above equation does not contain the derivative of the curvarutre, the equation contains only second order derivative of the metric tensor. As a classical theory, if we impose the initial conditions for metric tensor and its first derivative with respect to time, the metric tensor can be determined uniquely.

We now find a solution. We now assume the metric in the following form:

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 \sum_{i,j=1}^{d-1} \tilde{g}_{ij}dx^i dx^j. \quad (11)$$

Here \tilde{g}_{ij} : the metric tensor of the Einstein manifold defined by $\tilde{R}_{ij} = kg_{ij}$. Here \tilde{R}_{ij} is the Ricci tensor given by \tilde{g}_{ij} and k is a constant. For example, $k = d - 2$ corresponds to the $d - 1$ -dimensional unit sphere, $k = -(d - 2)$ to $d - 1$ -dimensional unit hyperboloid and $k = 0$ to a flat surface.

We consider the electromagnetic fields as a matter:

$$S_{\text{matter}} = -\frac{1}{4g^2} \int d^{d+1}x \sqrt{-g} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}, \quad (12)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (13)$$

Here A_μ is a vector potential (gauge field) and g is the gauge coupling. If $F_{tr} = -F_{rt}$ only depend on r and the other components of $F_{\mu\nu}$ vanish, one finds

$$F_{tr} = e^{\nu+\lambda} r^{1-d} Q \quad (14)$$

Here Q is a constant corresponding to the electric charge. Then the solution, which was first found in [10], appears

$$\begin{aligned} e^{2\nu} &= e^{-2\lambda} \\ &= \frac{1}{2c} \left[\frac{2ck}{d-2} + \frac{r^2}{\kappa^2(d-2)(d-3)} \right. \\ &\quad \pm \left\{ \frac{r^4}{\kappa^4(d-2)^2(d-3)^2} \right. \\ &\quad + \frac{4c\Lambda r^4}{d(d-1)(d-2)(d-3)} - \frac{2cQ^2 r^{6-2d}}{g^2(d-1)(d-3)} \\ &\quad \left. \left. - \frac{4cCr^{4-d}}{(d-1)(d-2)(d-3)} \right\}^{\frac{1}{2}} \right]. \end{aligned} \quad (15)$$

The radius $r = r_H$ of the event horizon, where $e^{2\nu} = 0$ or $e^{-2\lambda} = 0$, is given by

$$\begin{aligned} 0 &= \frac{(d-1)(d-3)}{d-2} ck^2 r_H^{d-4} + \frac{(d-1)k}{(d-2)\kappa^2} r_H^{d-2} \\ &\quad - \frac{\Lambda}{d} r_H^d + \frac{(d-2)Q^2}{2g^2} r_H^{2-d} + C, \end{aligned} \quad (16)$$

The Haking temperature T_H is also given by

$$\begin{aligned} 4\pi T_H &= (e^{2\nu})' \Big|_{r=r_H} \\ &= \frac{1}{2} \left(\frac{2ck}{d-2} + \frac{r_H^2}{\kappa^2(d-2)(d-3)} \right)^{-1} \\ &\quad \times \left[\frac{4kr_H}{\kappa^2(d-2)^2(d-3)} \right. \\ &\quad - \frac{8\Lambda r_H^3}{d(d-1)(d-2)(d-3)} - \frac{2Q^2 r_H^{5-2d}}{(d-1)g^2} \\ &\quad \left. - \frac{2(d-4)Cr_H^{3-d}}{(d-1)(d-2)(d-3)} \right]. \end{aligned} \quad (17)$$

Especially for $d = 4$, we have

$$\begin{aligned} e^{2\nu} &= e^{-2\lambda} \\ &= \frac{1}{2c} \left[ck + \frac{r^2}{2\kappa^2} \right. \\ &\quad \pm \left\{ \frac{r^4}{4\kappa^4} + \frac{c\Lambda r^4}{6} - \frac{2cQ^2}{3g^2 r^2} - \frac{2cC}{3} \right\}^{\frac{1}{2}} \right]. \end{aligned} \quad (18)$$

When r is large, the obtained solution behaves as

$$\begin{aligned} e^{2\nu} &= e^{-2\lambda} \\ &= \frac{1}{2c} \left[\frac{r^2}{2\kappa^2} \left(1 \pm \sqrt{1 + \frac{2c\Lambda\kappa^4}{3}} \right) + ck \right. \\ &\quad \left. \mp \frac{2\kappa^2 cC}{3r^2 \sqrt{1 + \frac{2c\Lambda\kappa^4}{3}}} \right] \end{aligned}$$

$$\mp \frac{2cQ^2}{3g^2 r^4 \sqrt{1 + \frac{2c\Lambda\kappa^4}{3}}} + \mathcal{O}(r^{-4}) \Big]. \quad (19)$$

We compare the above behavior with the Reissner-Nordström-AdS solution in the Einstein gravity:

$$e^{2\nu_{\text{RNAdS}}} = e^{-2\lambda_{\text{RNAdS}}} = \frac{r^2}{l^2} + \frac{k}{2} - \frac{\mu}{r^2} + \frac{q^2}{r^4}, \quad (20)$$

and we find

$$\begin{aligned} \frac{1}{l^2} &= \frac{1}{4c\kappa^2} \left(1 \pm \sqrt{1 + \frac{2c\Lambda\kappa^4}{3}} \right), \\ \mu &= \pm \frac{\kappa^2 C}{3\sqrt{1 + \frac{2c\Lambda\kappa^4}{3}}}, \\ q^2 &= \mp \frac{Q^2}{3g^2 \sqrt{1 + \frac{2c\Lambda\kappa^4}{3}}}, \end{aligned} \quad (21)$$

that is

$$\begin{aligned} \Lambda &= -\frac{12}{\kappa^2 l^2} + \frac{24c}{l^4}, \quad C = \mu \left(\frac{12c}{l^2} - \frac{3}{\kappa^2} \right), \\ \frac{Q^2}{g^2} &= 3q^2 \left(1 - \frac{4c\kappa^2}{l^2} \right). \end{aligned} \quad (22)$$

4. Hawking-Page phase transition in R^2 -gravity

Since the black hole has temperature, one can consider the thermodynamics of the spacetime. The theory with finite temperature for an action $S(\phi)$ is given by Wick-rotating the time coordinate ($t \rightarrow it$) and imposing the periodic boundary condition for t with the period β : $Z(\beta) = \int [d\phi] e^{S(\phi)}$. The period β can be regarded as the inverse of the temperature T : $\beta = \frac{1}{k_B T}$. Here k_B is the Boltzmann constant and we put $k_B = 1$ in the following. Then $Z(\beta)$ can be regarded as the thermodynamical partition function. In the WKB approximation, by substituting the classical solution $\phi = \phi_{\text{class}}$ into the action, we obtain the free energy F by $F = -T \ln Z(\beta) = -TS(\phi_{\text{class}})$. In the following, we concentrate on the $D = d + 1 = 5$ dimensional asymptotically AdS spacetime.

Before going to the R^2 -gravity, we consider the case of Einstein gravity without matter. When the spacetime metric is given by

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{-2\nu(r)} dr^2 + r^2 \sum_{i,j=1}^3 \tilde{g}_{ij} dx^i dx^j, \quad (23)$$

we find

$$\begin{aligned} F &= -T \ln Z(\beta) = -TS_{\text{Einstein}} \\ S_{\text{Einstein}} &= \int d^5 x \sqrt{g} \left\{ \frac{1}{\kappa^2} R - \Lambda \right\} \\ &= -\frac{8V_3}{T l^2 \kappa^2} \int_{r_H}^{\infty} dr r^3, \end{aligned}$$

$$\Lambda = -\frac{12}{l^2 \kappa^2}, \quad V_3 = \int d^3 x \sqrt{\tilde{g}}. \quad (24)$$

Then S_{Einstein} diverges. The regularization of the divergence is given by cutting off the integral at a large radius $r = r_{\text{max}}$ and subtracting the solution of pure but finite temperature AdS:

$$S_{\text{reg}} = -\frac{8V_3}{T\kappa^2 l^2} \left\{ \int_{r_H}^{r_{\text{max}}} dr r^3 - e^{\nu(r=r_{\text{max}}) - \rho(r=r_{\text{max}}; \mu=0)} \int_0^{r_{\text{max}}} dr r^3 \right\} \quad (25)$$

The factor $e^{\nu(r=r_{\text{max}}) - \rho(r=r_{\text{max}}; \mu=0)}$ is chosen so that the proper length of the circle which corresponds to the period $\beta = \frac{1}{T_H}$ in the Euclidean time at $r = r_{\text{max}}$ coincides with each other in the two solutions. Especially for $k = 2$, we find in the limit of $r_{\text{max}} \rightarrow \infty$

$$F = -\frac{V_3}{\kappa^2} r_H^2 \left(\frac{r_H^2}{l^2} - 1 \right). \quad (26)$$

When $\frac{r_H^2}{l^2} > 1$, the free energy becomes negative: $F < 0$, which tells that black hole spacetime is more stable than the pure AdS spacetime. That is, large black holes are stable but small ones are unstable and there is a critical point at $\frac{r_H^2}{l^2} = 1$, which is the famous Hawking-Page phase transition [11].

In $c = 0$ R^2 -gravity case, we find

$$F = -\frac{V_3}{8} r_H^2 \left(\frac{r_H^2}{l^2} - \frac{k}{2} \right) \left(\frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right), \quad (27)$$

which is similar to that in the Einstein gravity case (26) but there is a critical point (surface) at

$$\frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} = 0. \quad (28)$$

Therefore when $\frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} < 0$, small black hole is stable.

In case of the Gauss-Bonnet-Maxwell black hole case, we find

$$F = -V_3 \left\{ -3 \left(\frac{2c}{l^4} - \frac{1}{\kappa^2 l^2} \right) r_H^4 - 2\pi T_H r_H \left(\frac{r_H^2}{\kappa^2} - 12c \right) \right\}. \quad (29)$$

Since we have two independent parameters μ and Q^2 , r_H can be independent of T_H . Then the critical line, where $F = 0$, is given by

$$T_H = T_c \equiv \frac{3}{2\pi} \left(\frac{r_H^2}{\kappa^2} - 6ck \right)^{-1} \left(\frac{1}{\kappa^2 l^2} - \frac{2c}{l^4} \right) r_H^3. \quad (30)$$

If $T_H > T_c$ and $r_H^2 > 12c\kappa$ (or, $T_H < T_c$ and $r_H^2 < 12c\kappa$), pure anti-de Sitter spacetime is more stable than the black hole spacetime. On the other hand, if $T_H < T_c$ and $r_H^2 > 12c\kappa$ (or, $T_H > T_c$ and $r_H^2 < 12c\kappa$), vice

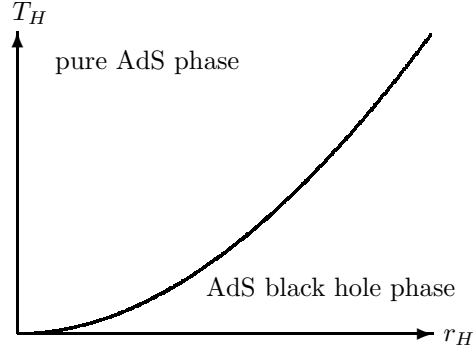


Figure 1: The phase diagrams when $\epsilon < 0$.

versa. In order to simplify the expressions, we (re)define the following parameters and the variables:

$$\epsilon \equiv \frac{c\kappa^2}{l^2}, \quad r_H \rightarrow lr_H, \quad T_H \rightarrow \frac{T_H}{l}. \quad (31)$$

Then the critical temperature T_c is expressed as

$$T_c = \frac{3(1 - 2\epsilon)r_H^3}{2\pi(r_H^2 - 12\epsilon)}. \quad (32)$$

From the above expression (32), one gets:

- When $\epsilon < 0$ ($< \frac{1}{2}$), T_c is always positive. Then if $T_H > T_c$ ($T_H < T_c$), the pure AdS (black hole) spacetime is more stable than black hole (pure AdS) spacetime.
- When $0 < \epsilon < \frac{1}{2}$, the critical temperature T_c is positive when $r_H^2 > 12\epsilon$ and there is a critical line, where if $T_H > T_c$ ($T_H < T_c$), the pure AdS (black hole) spacetime is more stable than the black hole (pure AdS) spacetime. When $r_H^2 < 12\epsilon$, T_c is negative, then the black hole spacetime is always stable.
- When $\epsilon > \frac{1}{2}$, if $r_H^2 > 12\epsilon$, T_c is negative and the pure AdS spacetime is more stable than the black hole spacetime. If $r_H^2 < 12\epsilon$, T_c is positive and if $T_H > T_c$ ($T_H < T_c$), the black hole (pure AdS) spacetime is more stable than the pure AdS (black hole) spacetime.

The conceptual (not-exact) Hawking-Page phase diagrams are given by Figures 1, 2 and 3.

5. Negative Entropy?

In this section, based on the entropy, we investigate the relation between Schwarzschild-de Sitter black hole and Schwarzschild-*anti* de Sitter black hole.

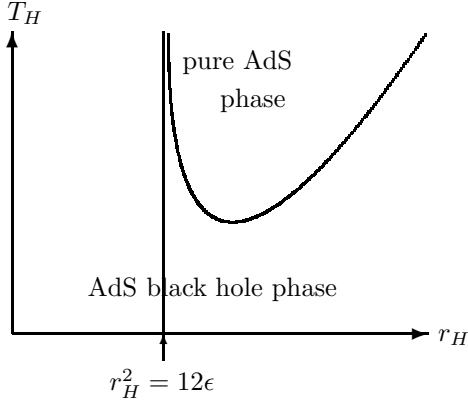


Figure 2: The phase diagrams when $0 < \epsilon < \frac{1}{2}$.

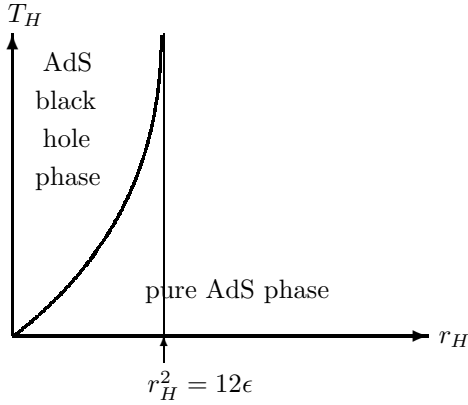


Figure 3: The phase diagrams when $\epsilon > \frac{1}{2}$.

In the spacetime with black hole, we put a boundary, which is a surface with constant r . On the boundary, one takes an action, which makes the variational principle well-defined (like Gibbons-Hawking term) and further makes the total action finite. Then we can define an energy-momentum tensor on the boundary and we can determine the mass M ($d = 4$):

$$\begin{aligned} E &= M \\ &= \frac{3l^2}{16} V_3 \left(\frac{1}{\kappa^2} - \frac{40a}{l^2} - \frac{8b}{l^2} - \frac{4c}{l^2} \right) \\ &\quad \times \left(k^2 + \frac{16\mu}{l^2} \right). \end{aligned} \quad (33)$$

M is thermodynamical energy E [12].

By using the thermodynamical relation $dS = \frac{dE}{T}$, in $c = 0$ case, one arrives at

$$S = \int \frac{dE}{T_H} = \frac{V_3 \pi r_H^3}{2} \left(\frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right) + S_0. \quad (34)$$

Here S_0 is a constant of the integration. Then there

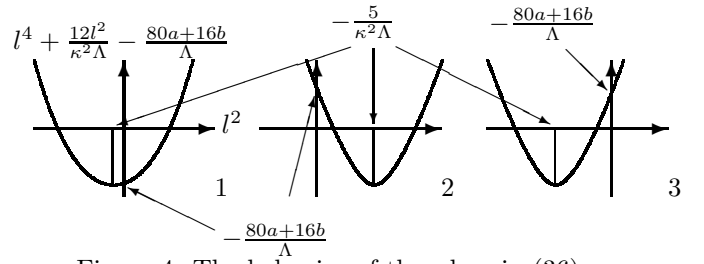


Figure 4: The behavior of the r.h.s. in (36).

might appear a natural question, that is, when

$$\frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} < 0, \quad (35)$$

is entropy negative? Even if $S_0 \neq 0$, the entropy becomes negative for the large (large r_H) black hole. We should note that l^2 ($d = 4$) is given by solving Eq.(6) with $d = 4$, which can be rewritten as

$$\begin{aligned} 0 &= l^4 + \frac{12l^2}{\kappa^2 \Lambda} - \frac{80a + 16b}{\Lambda} \\ &= \left(l^2 + \frac{5}{\kappa^2 \Lambda} \right)^2 - \frac{36}{\kappa^4 \Lambda^2} - \frac{80a + 16b}{\Lambda}. \end{aligned} \quad (36)$$

Then if

$$\frac{36}{\kappa^4 \Lambda^2} + \frac{80a + 16b}{\Lambda} \geq 0, \quad (37)$$

there are real solution of l^2 :

$$\frac{1}{l^2} = \frac{\frac{6}{\kappa^2} \pm \sqrt{\frac{36}{\kappa^4} + \Lambda(80a + 16b)}}{80a + 16b}. \quad (38)$$

Then from Figure 4, we find

1. When $80a + 16b > 0$ and $\Lambda > 0$, or $80a + 16b < 0$ and $\Lambda < 0$, one of the solutions of l^2 is positive but another is negative. Then there are two kind of solution, one is asymptotically anti-de Sitter and another is asymptotically de Sitter.
2. When $80a + 16b > 0$ and $\Lambda < 0$, l^2 is always positive. Both of two solutions express the asymptotically anti-de Sitter spacetime.
3. When $80a + 16b < 0$ and $\Lambda > 0$, $l^2 < 0$. Both of two solutions correspond to asymptotically de Sitter spacetime.

The expression of the entropy (34) can be applied even for asymptotically de Sitter spacetime by putting $l^2 = -l_{\text{dS}}^2 < 0$. By the explicit expression of l^2 via Λ , a , b , and κ^2 , we obtain

$$S = \frac{V_3 \pi r_H^3}{2} \left(-\frac{16}{\kappa^2} \mp \sqrt{\frac{36}{\kappa^4} + \Lambda(80a + 16b)} \right). \quad (39)$$

If we assume the entropy should be positive, we need to choose the lower sign $+$ in \pm since the entropy always

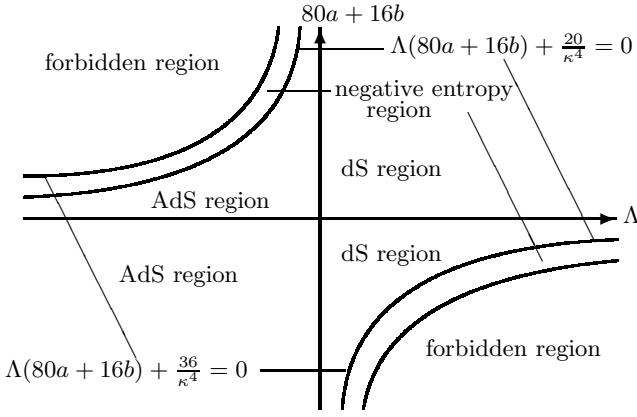


Figure 5: The phase diagram of the spacetime with respect to the parameters.

becomes negative if we choose the upper sign $-$. Even if one chooses the lower sign $+$, the entropy is not always positive but it becomes positive when

$$\frac{20}{\kappa^4 \Lambda^2} + \frac{80a + 16b}{\Lambda} \geq 0. \quad (40)$$

Then we obtain the diagram in Figure 5.

In the Gauss-Bonnet case, we obtain

$$E = M = \frac{3l^2}{16} V_3 \left(\frac{1}{\kappa^2} - \frac{12c}{l^2} \right) \left(k^2 + \frac{16\mu}{l^2} \right) \quad (41)$$

and therefore

$$\mathcal{S} = \frac{V_3}{\kappa^2} \left(\frac{1 - 12\epsilon}{1 - 4\epsilon} \right) (4\pi r_H^3 + 24\epsilon k \pi r_H) + \mathcal{S}_0, \quad (42)$$

Here $\epsilon \equiv \frac{c\kappa^2}{l^2}$ and the length parameter l^2 is given by

$$\frac{1}{l^2} = \frac{1}{4c\kappa^2} \left(1 \pm \sqrt{1 + \frac{2c\Lambda\kappa^4}{3}} \right). \quad (43)$$

Then in order that l^2 is a real number, we have

$$c\Lambda \geq -\frac{3}{2\kappa^4}. \quad (44)$$

Then we find

1. When $c\Lambda > 0$, there exist two kind of solutions, one expresses asymptotically AdS spacetime and another expresses the asymptotically dS spacetime.
2. When $c > 0$ and $\Lambda < 0$, both of two solutions expresses asymptotically AdS spacetime.
3. When $c < 0$ and $\Lambda > 0$, both of solutions correspond to asymptotically dS spacetime.

Substituting the expression of l^2 , we obtain

$$\mathcal{S} = \frac{V_3}{\kappa^2} \left(3 \pm \frac{2}{\sqrt{1 + \frac{2c\Lambda\kappa^4}{3}}} \right) (4\pi r_H^3 + 24\epsilon k \pi r_H) + \mathcal{S}_0. \quad (45)$$

Therefore when

$$c\Lambda > -\frac{5}{6\kappa^4}, \quad (46)$$

the entropy is always positive for both of solutions corresponding to the \pm sign. Especially when $c\Lambda > 0$, both of solutions expressing asymptotically AdS and dS spacetime. Then we cannot exclude one of the solution from the viewpoint of the entropy.

6. Application to the brane cosmology

We now consider application of the Gauss-Bonnet gravity to the brane cosmology. Here we consider $Q = 0$ case for the simplicity. In the bulk black hole spacetime in the Gauss-Bonnet gravity, the motion of the brane can be described by the Friedmann-like equation. One may employ the method developed in Ref.[5, 6, 13] to derive the Friedmann equation. Then we obtain

$$H^2 = \frac{\mathcal{G}^2}{\mathcal{H}^2} - \frac{X(a)}{a^2} \equiv f(a), \quad (47)$$

where

$$\begin{aligned} \mathcal{G} &= 4\eta \pm \frac{12X^{1/2}}{Y^{3/2}} \{16\epsilon^2 \tilde{\mu}^2 (4\epsilon - 1)^2 a^{-3}\} \\ \eta &\equiv \frac{6}{\kappa_g^2 l} (12\epsilon - 1). \end{aligned} \quad (48)$$

and

$$\begin{aligned} \mathcal{H} &= -\frac{48}{\kappa_g^2} \mp \frac{24}{Y^{3/2}} \left(5(2\epsilon \tilde{\mu} (4\epsilon - 1))^2 a^{-2} \right. \\ &\quad \left. + 6 \left(\frac{(4\epsilon - 1)^2}{4\kappa_g^4} \right)^2 a^6 - \frac{9(4\epsilon - 1)^3 \epsilon \tilde{\mu}}{2\kappa_g^4} a^2 \right), \quad (49) \\ X &\equiv \frac{k}{2} + \frac{a^2}{4\epsilon l^2} \pm \frac{Y^{1/2} \kappa_g^2}{2\epsilon l^2}, \\ Y &\equiv -2\epsilon \tilde{\mu} (4\epsilon - 1) + \frac{(4\epsilon - 1)^2}{4\kappa_g^4} a^4. \end{aligned} \quad (50)$$

Here we have defined the rescaled parameters: $\epsilon \equiv \frac{c\kappa^2}{l^2}$ and $\tilde{\mu} \equiv \frac{l^2 \mu}{\kappa_g^4}$. The standard FRW equations for 4-dimensions can be written as

$$\begin{aligned} H^2 &= \frac{8\pi G}{3} \rho - \frac{k}{2a^2}, \\ \dot{H} &= -4\pi G (\rho + p) + \frac{k}{2a^2}. \end{aligned} \quad (51)$$

Here, \cdot is the derivative with respect to cosmological time. Then ρ and p are

$$\rho = \frac{3}{8\pi G} \left(f(a) + \frac{k}{2a^2} \right) \quad (52)$$

$$p = -\frac{1}{8\pi G} \left(\frac{k}{2a^2} + af'(a) + 3f(a) \right), \quad (53)$$

where \cdot denotes the derivative with respect to a . Similarly, FRW equations from the bulk dS BH can be constructed (see [13]).

In case of $a \rightarrow \infty$, we obtain

$$\rho = \frac{3}{8\pi G} \left(\frac{1}{l^2} (12\epsilon - 1)^2 (2 \pm 3(4\epsilon - 1))^{-2} \right. \\ \left. - \frac{1}{4\epsilon l^2} \mp \frac{4\epsilon - 1}{4\epsilon l^2} \left(1 - \epsilon \tilde{\mu} \frac{4\kappa_g^4}{(4\epsilon - 1)a^4} \right) \right) \\ p = -\frac{1}{8\pi G} \left(\frac{3}{l^2} (12\epsilon - 1)^2 (2 \pm 3(4\epsilon - 1))^{-2} \right. \\ \left. - \frac{3}{4\epsilon l^2} \mp \frac{3(4\epsilon - 1)}{4\epsilon l^2} \mp \frac{1}{l^2} \tilde{\mu} \kappa_g^4 a^{-4} \right). \quad (54)$$

$$- \frac{3}{4\epsilon l^2} \mp \frac{3(4\epsilon - 1)}{4\epsilon l^2} \mp \frac{1}{l^2} \tilde{\mu} \kappa_g^4 a^{-4} \quad (55)$$

When we choose the upper sign of \pm and \mp , the expressions in (54) and (55) are simplified as

$$\rho = \frac{3}{8\pi G} \frac{\tilde{\mu} \kappa_g^4}{l^2 a^4}, \quad p = \frac{1}{8\pi G} \frac{\tilde{\mu} \kappa_g^4}{l^2 a^4}. \quad (56)$$

Then the energy-momentum tensor is traceless ($T^\mu_\mu = \rho - 3p = 0$.) and there should be CFT on the brane. On the other hand, if we choose the lower signs, we find the FRW equations with the cosmological term:

$$H^2 = \frac{8\pi G}{3} \rho_m - \frac{k}{2a^2} + \frac{\Lambda}{3}, \\ \dot{H} = -4\pi G (\rho + p) + \frac{k}{2a^2}, \quad (57)$$

We can divide ρ and p into the sums of the contributions from the matter and those from the cosmological term:

$$\rho = \rho_m + \rho_0, \quad p = p_m - \rho_0, \quad \rho_0 = \frac{\Lambda}{8\pi G}. \quad (58)$$

Then we obtain

$$\rho_0 = \frac{3}{8\pi G} \left\{ \frac{1}{l^2} (12\epsilon - 1)^2 (5 - 12\epsilon)^{-2} \right. \\ \left. - \frac{1 - 2\epsilon}{2\epsilon l^2} \right\}, \\ \rho_m = -\frac{3}{8\pi G} \frac{\tilde{\mu} \kappa_g^4}{l^2 a^4}, \quad p_m = -\frac{1}{8\pi G} \frac{\tilde{\mu} \kappa_g^4}{l^2 a^4}. \quad (59)$$

On the other hand, the energy-momentum tensor $T^\mu_{\mu\nu}$ of the matter is again traceless,

$$T^\mu_\mu = \rho_m - 3p_m = 0. \quad (60)$$

Then the matter field should be CFT again.

We now consider the limit of $a \rightarrow 0$ when $-2\epsilon\tilde{\mu}(4\epsilon - 1) > 0$.

$$\rho = -3p \\ = \frac{3}{8\pi G} \left\{ \frac{2k}{25} \right. \\ \left. \mp \frac{21}{25} \frac{\kappa_g^2}{2\epsilon l^2} (-2\epsilon\tilde{\mu}(4\epsilon - 1))^{1/2} \right\} a^{-2}. \quad (61)$$

Then the energy-momentum tensor is not traceless. Since $\rho, p \sim a^{-2}$, this limit corresponds to the curvature dominant case. We now rewrite the FRW equation in the following form:

$$H^2 = \frac{8\pi G}{3} \left(\rho_m - \frac{3}{8\pi G} \frac{k}{2a^2} \right) + \frac{\Lambda}{3} \\ = \frac{8\pi G}{3} \tilde{\rho} - \frac{\tilde{k}}{2a^2} + \frac{\Lambda}{3}. \quad (62)$$

Here $\tilde{\rho}$ and \tilde{k} can be regarded as the effective energy density and effective k :

$$\tilde{\rho} = \rho_m - \frac{3}{8\pi G} \frac{k}{2a^2}, \quad \tilde{k} = 0. \quad (63)$$

When k is large, $\tilde{\rho}$ behaves as a^{-2} .

When $-2\epsilon\tilde{\mu}(4\epsilon - 1) < 0$, X in (50), \mathcal{H} in (49) and \mathcal{G} in (48) become complex at $a = 0$, which may tell that there is a minimum of a given by $Y = 0$:

$$a = a_{\min} \equiv \left(\frac{8\epsilon\tilde{\mu}}{(4\epsilon - 1)} \right)^{\frac{1}{4}} \kappa_g. \quad (64)$$

When $a = a_{\min}$, we have

$$\rho = -3p \\ = \frac{3}{8\pi G} \left\{ \frac{2k}{81} \left(\frac{2\epsilon\tilde{\mu}}{(4\epsilon - 1)} \right)^{-\frac{1}{2}} (2\kappa_g^2)^{-1} \right. \\ \left. - \frac{77}{81} \frac{1}{4\epsilon l^2} \right\}. \quad (65)$$

Then by dividing them into the contributions from the cosmological term and the matter, we find

$$\rho_0 = -\frac{3}{8\pi G} \frac{77}{81} \frac{1}{4\epsilon l^2}, \\ \rho_m = \frac{3}{8\pi G} \frac{2k}{81} \left(\frac{2\epsilon\tilde{\mu}}{(4\epsilon - 1)} \right)^{-\frac{1}{2}} (2\kappa_g^2)^{-1} \\ p_m = -\frac{1}{8\pi G} \frac{2k}{81} \left(\frac{2\epsilon\tilde{\mu}}{(4\epsilon - 1)} \right)^{-\frac{1}{2}} (2\kappa_g^2)^{-1}. \quad (66)$$

As a special case, we can consider the case of $\epsilon = 1/4$. Since

$$f(a) = -\frac{k}{2a^2}, \quad (67)$$

and $f(a) = H^2 \geq 0$, we find $k \leq 0$. In this case, the energy density and the pressure vanish: $\rho = 0, p = 0$.

If one put $\epsilon = 1/4 - \delta^2$ and δ is small, the energy density and the pressure are given by

$$\begin{aligned}\rho &= -3p \\ &= \pm \frac{3}{8\pi G} \frac{\kappa_g^2}{l^2} \left\{ \frac{2l}{a^3} \left(\frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} - \frac{7}{a^2} \right\} (2\tilde{\mu})^{1/2} \delta.\end{aligned}\quad (68)$$

Then in the limit of $a \rightarrow 0$, we have $|\rho| \gg |p|$ and $\rho \sim a^{-3}$, which corresponds to “dust”. On the other hand, when $a \rightarrow \infty$, we find (if $k \geq 0$) $\rho, p \sim a^{-2}$, which tells curvature dominant. When $k = -2 < 0$, a has a minimum $a = l$.

More applications to the cosmology are given in [13, 14]. Especially the solution describing the bounce universe has been found.

7. Summary

We have obtained exact solutions in R^2 -gravity when the action does not contain the term of the square of the Riemann tensor ($c = 0$) and when the R^2 -terms are the Gauss-Bonnet combination with the electromagnetic fields. We have also investigated the thermodynamics of the R^2 -gravity and we have found that the entropy often becomes negative. The application to the brane cosmology is interesting and given in [13, 14].

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